Weak Measurement in Quantum Mechanics

Electronic Structure Discussion Group

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At first sight the claim made by the theory of weak measurement, proposed in 1988 by Aharonov, Albert and Vaidman (AAV), seems to contradict the basics of quantum mechanics:

Quantum mechanical measurement of observables can, under certain circumstances, yield values which are not eigenvalues of the observables and can even lie outside (and far away) from the spectrum of eigenvalues.

However we will see that this is merely a (possibly quite useful) extension of Quantum Mechanics rather than a contradiction to it.

(Almost) Conventional Quantum Measurement

In conventional quantum-mechanical measurement an instantaneous interaction is assumed. If we introduce a variable $\hat{q}$, corresponding to the "pointer position" of our measuring device, and its conjugate momentum $\hat{p}$, then the time dependent interaction Hamiltonian H_i for a measurement at time t_0 may be written as:

$$\hat{H}_i = \delta(t - t_0)\hat{p}\hat{A}$$

Now lets assume an initial Gaussian distribution of width Δ of the pointer position and momentum states so that the initial state of the pointer can be written as:

$$|\phi\rangle = \int \exp(\frac{-p^2}{4\Delta^2})|p\rangle dp = \int \exp(-\Delta^2 q^2)|q\rangle dq$$

Measurement Evolution

Hence in the event of measurement the combined state of quantum system and apparatus $|\Psi\rangle|\phi\rangle$ evolves as follows:

$$|\Psi\rangle|\phi\rangle \to \exp(-i\int \hat{H}_i dt)|\Psi\rangle|\phi\rangle = \exp(-i\hat{p}\hat{A})|\Psi\rangle|\phi\rangle$$

Expanding $|\Psi\rangle$ in terms of the eigenbasis $|a_k\rangle$ of operator $\hat{A}$, i.e. $|\Psi\rangle = \sum_k \alpha_k|a_k\rangle$, we get:

$$|\Psi\rangle|\phi\rangle \to \sum_k \alpha_k \exp(-i\hat{p}a_k)|a_k\rangle|\phi\rangle$$

Measurement Evolution (cont.)

Using the p representation of ϕ we get:

$$|\Psi\rangle|\phi\rangle \rightarrow \sum_k \alpha_k \int \exp(-ipa_k)\ \exp(\frac{-p^2}{4\Delta^2})|a_k\rangle|p\rangle dp$$

We can convert to q representation by inserting $I = \int dq|q\rangle\langle q|$ and noting that $\langle p|q\rangle = \exp(ipq)$:

$$|\Psi\rangle|\phi\rangle \rightarrow \sum_k \alpha_k \int \exp(-\Delta^2(q-a_k)^2)|a_k\rangle|q\rangle dq$$

which corresponds to a set of Gaussians of width $\frac{1}{2\Delta}$, centred around the eigenvalues a_k. A weak measurement simply means that we let Δ become large compared to the spacing of the eigenvalues a_k.

Where does the pointer point?

The probability distribution of the pointer is given by the squared modulus of the overlap between the total wavefunction and $\langle q|$:

$$P(q) = |\langle q| \sum_k \alpha_k \int \exp(-\Delta^2 (q - a_k)^2) |a_k\rangle |q\rangle dq|^2$$

$$= \sum_k |\alpha_k|^2 \exp(-2\Delta^2 (q - a_k)^2)$$

Post-Selection

Post-selection means that after the weak measurement we perform a strong measurement on the quantum system and select one of the outcomes $|\Psi_f\rangle = \sum_k \alpha'_k |a_k\rangle$. (One example would be the selection of one of the beams emerging from the Stern-Gerlach apparatus.) Hence the final state of the measurement apparatus would be:

$$|\phi_f\rangle = \langle\Psi_f|\exp(-i\int \hat{H}_i dt)|\Psi\rangle|\phi\rangle$$

$$= \sum_k \alpha_k \alpha'^*_k \int \exp(-\Delta^2(q-a_k)^2)|q\rangle dq$$

which is a sum of Gaussians with *complex* coefficients.

Where does a post-selected pointer point?

Again the probability distribution of the pointer is given by the squared modulus of the overlap between the total wavefunction after post-selection and $\langle q|$:

$$P(q)_{post} = |\langle q| \sum_k \alpha_k \alpha_k'^* \int \exp(-\Delta^2 (q - a_k)^2) |q\rangle dq|^2$$

$$= | \sum_k \alpha_k \alpha_k'^* \exp(-\Delta^2 (q - a_k)^2)|^2$$

At first this might look quite similar to the non-post-selected pointer, but it is actually $very$ different:

$$P(q)_{non-post} = \sum_k |\alpha_k|^2 \exp(-2\Delta^2 (q - a_k)^2)$$

How do Gaussians add?

For simplicity assume two eigenvalues $a_1 = 1$ and $a_2 = 2$ with coefficients $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = -\frac{\sqrt{3}}{2}$, so:

$$|\Psi\rangle = \frac{1}{2}|1\rangle - \frac{\sqrt{3}}{2}|2\rangle$$

From above we can see that without post-selection our probability distribution of q is:

$$P(q) = \sum_k |\alpha_k|^2 \exp(-2\Delta^2(q - a_k)^2)$$

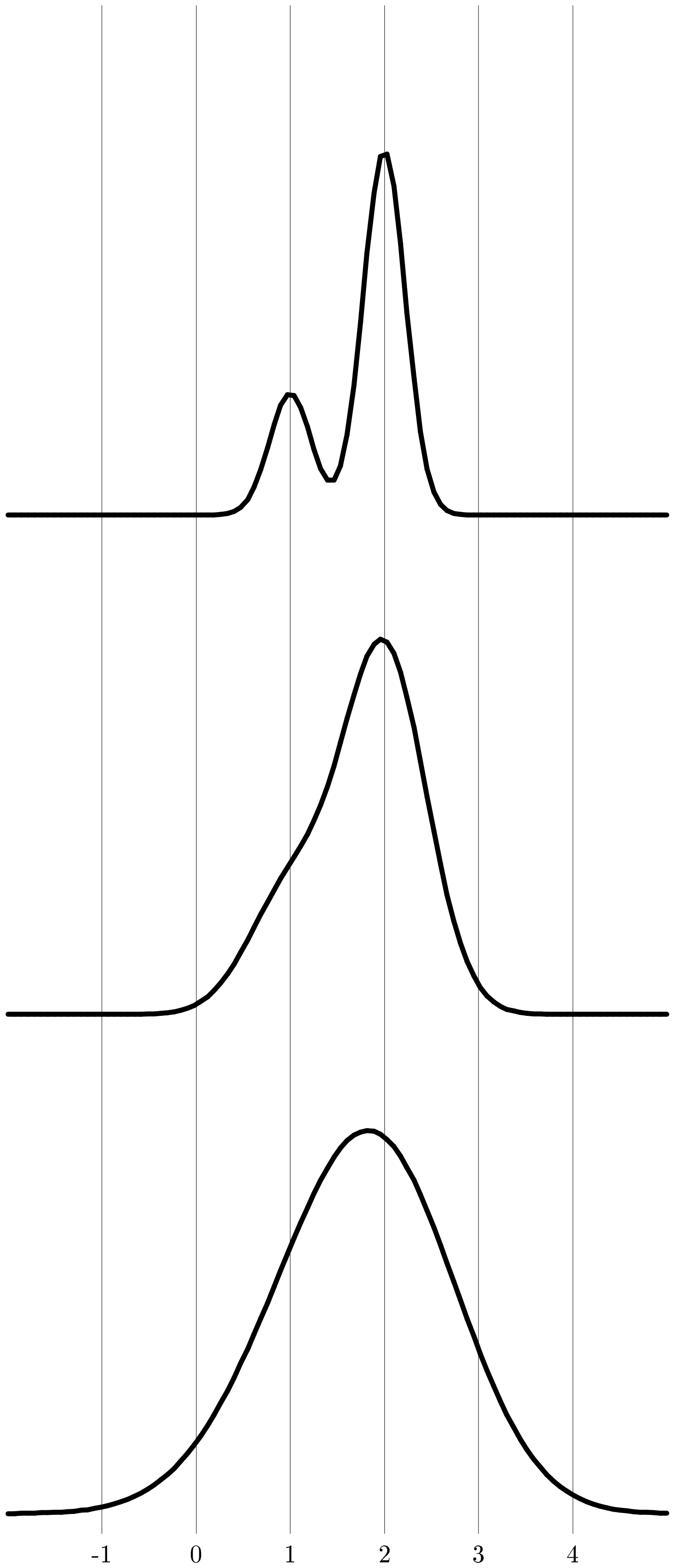
$$= \frac{1}{4}\exp(-2\Delta^2(q-1)^2) + \frac{3}{4}\exp(-2\Delta^2(q-2)^2)$$

width:

$\frac{1}{2\Delta} = \sqrt{0.0125}$

$\frac{1}{2\Delta} = \sqrt{0.05}$

$\frac{1}{2\Delta} = \sqrt{0.175}$



How do Gaussians add...
...with postselection?

If we post-select, then we are effectively partially collapsing the state. The probability distribution is not normalized anymore, as we are only considering events where the post-selection has been successful. Consider the post-selected state (where ϵ is small):

$$|\Psi_f\rangle = (\frac{\sqrt{3}}{2} + \epsilon)|1\rangle + (\frac{1}{2} - \sqrt{3}\epsilon)|2\rangle$$

which is almost orthogonal to $|\Psi\rangle$. Then

$$P(q)_{post} = |\sum_k \alpha_k \alpha_k'^* \exp(-\Delta^2(q-a_k)^2)|^2$$

$$= |(\frac{\sqrt{3}}{4} + \frac{\epsilon}{2})\exp(-\Delta^2(q-1)^2)$$
$$-(\frac{\sqrt{3}}{4} - \frac{3\epsilon}{2})\exp(-\Delta^2(q-2)^2)|^2$$

width:

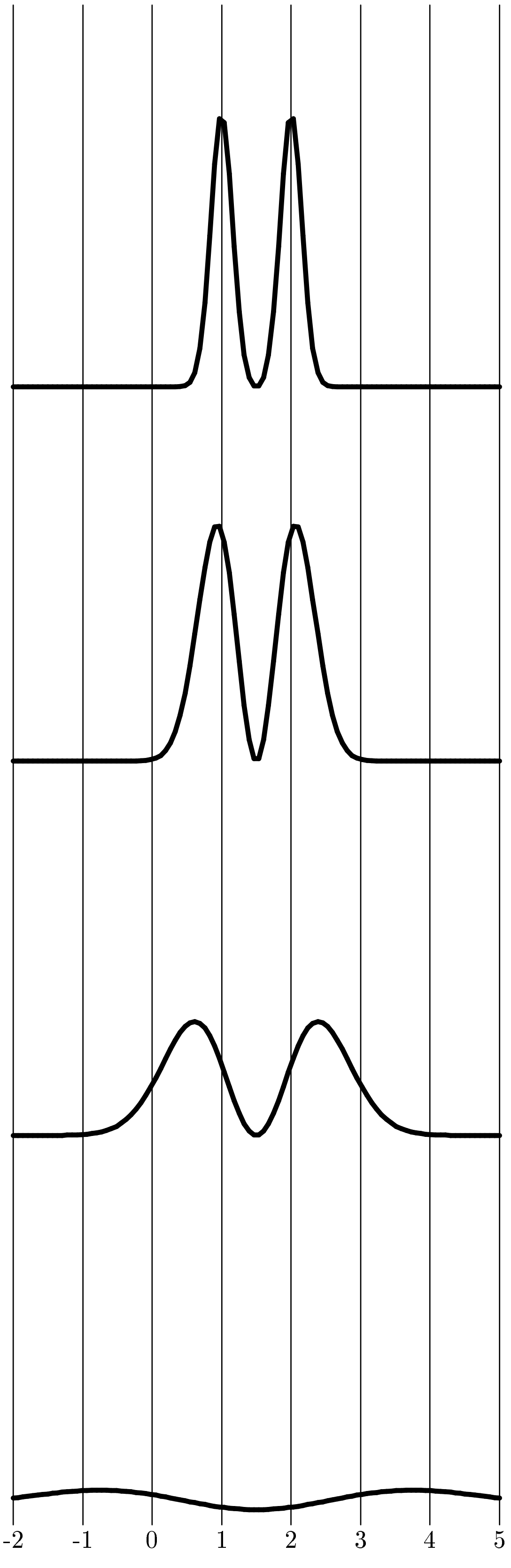
$\epsilon = 10^{-5}$

$\frac{1}{2\Delta} = \sqrt{0.025}$

$\frac{1}{2\Delta} = \sqrt{0.1}$

$\frac{1}{2\Delta} = \sqrt{0.35}$

$\frac{1}{2\Delta} = \sqrt{2.5}$



Why is this relevant?

A very good question. After all, we have just introduced errors into our measurement apparatus, so what is special about measuring values outside the eigenspectrum?

The answer is that weak values (i.e. the pointer positions in the case of large Δ and appropriate post-selection) are **physically consistent**.

This means that for instance, if we perform a weak measurement of the kinetic energy of an electron inside a potential well and post-select it to be inside the barrier (which it could never be in a strong measurement), then we retrieve the 'correct' value of negative kinetic energy.

Experimental Evidence

In 1990 Ritchie et al. performed a weak measurement using a Gaussian beam of light polarized at 45^o and then sent through a polarizing beamsplitter.

The two resulting beams were kept very close together ($\rightarrow$ overlapping Gaussians) and then post-selected with a polarization filter oriented very close to -45^o.

The photons that hit the detection screen after postselection were displaced by 120 times the beam width, relative to the axes of the beams.

Weak Values and Paradoxes

Weak values can also be used as a better language for explaining counterfactual paradoxes in quantum mechanics, although there is an ongoing philosophical debate about this.

Fact is that the formalism can be used to resolve Hardy's paradox, which is an elegant proof of the non-locality of quantum mechanics.

The weak values in this case are negative numbers of particles in the arms of overlapping interferometers.

References and Further Reading

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